STA 360/602L: Module 2.2

Operationalizing data analysis; selecting priors

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Outline

- Operationalizing data analysis
- Example: rare events
- Selecting priors and potential problems

Operationalizing data analysis

How should we approach data analysis in general?

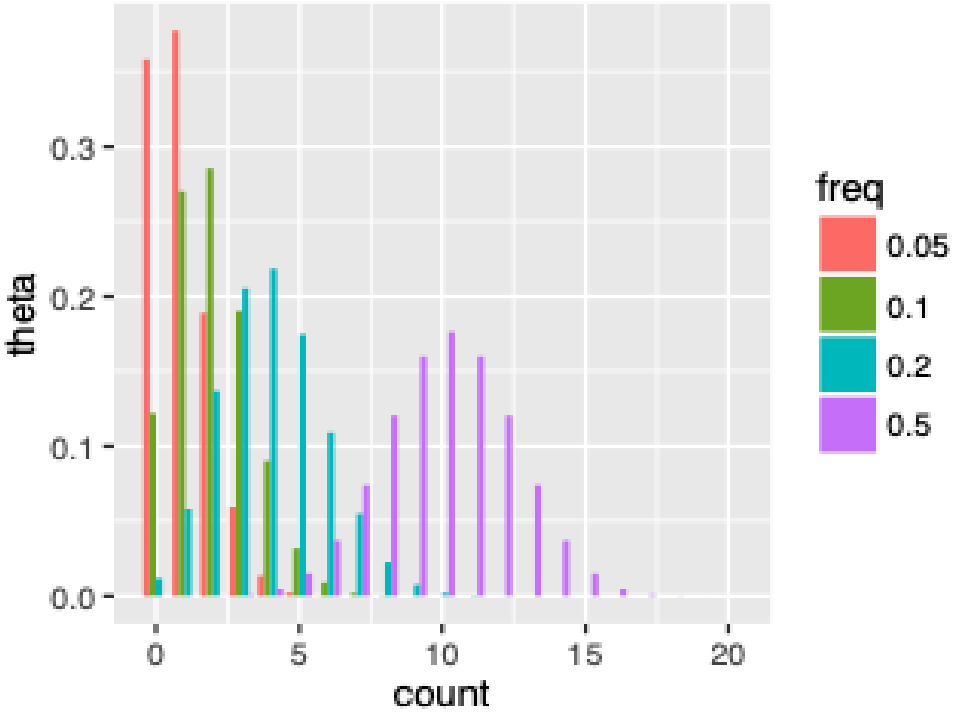
- Step 1. State the question.
- Step 2. Collect the data.
- Step 3. Explore the data.
- Step 4. Formulate and state a modeling framework.
- Step 5. Check your models.
- Step 6. Answer the question.

Example: rare events

- Step 1. State the question:
  - What is the prevalence of an infectious disease in a small city?
  - Why? High prevalence means more public health precautions are recommended.
- Step 2. Collect the data:
  - Suppose you collect a small random sample of 20 individuals.
- Step 3. Explore the data:
  - Let Y denote the unknown number of infected individuals in the sample.

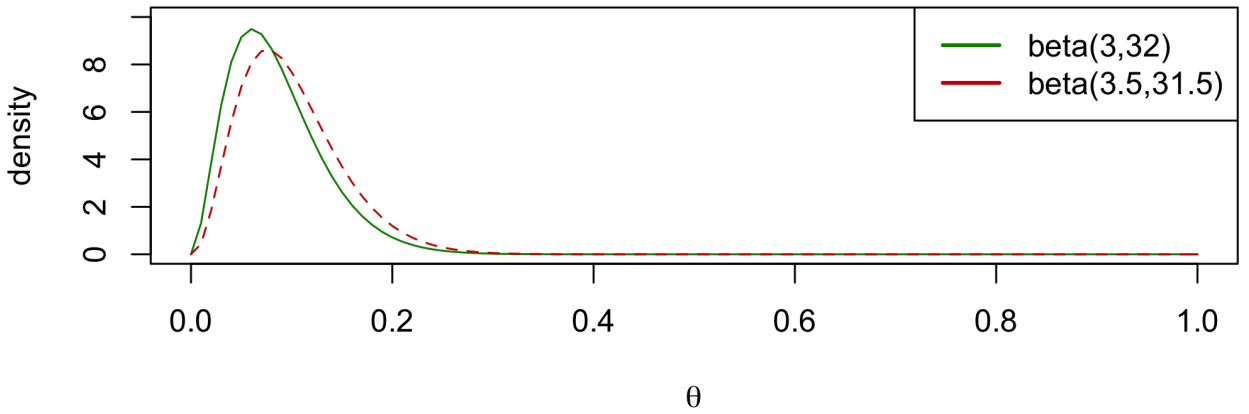
EXAMPLE: RARE EVENTS

- Step 4. Formulate and state a modeling framework:
  - Parameter of interest: θ is the fraction of infected individuals in the city.
  - Sampling model: a reasonable model for Y can be $\text{Bin}(20, \theta)$



Example: rare events

- Step 4. Formulate and state a modeling framework:
  - Prior specification: information from previous studies — infection rate in "comparable cities" ranges from 0.05 to 0.20 with an average of 0.10. So maybe a standard deviation of roughly 0.05?
  - What is a good prior? The **expected value** of θ close to 0.10 and the **standard deviation** close to 0.05.
  - Possible option: $\text{Beta}(3.5, 31.5)$ or maybe even $\text{Beta}(3, 32)$?



Quick beta-binomial recap

- Binomial likelihood:

$$p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

- $+$ Beta Prior:

$$\pi(\theta) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1} = \text{Beta}(a,b)$$

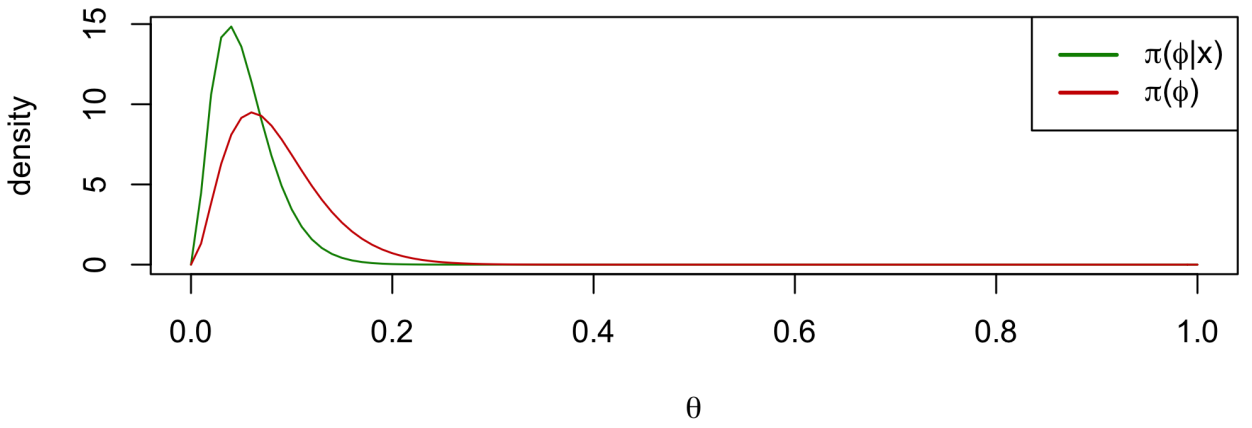
- $\Rightarrow$ Beta posterior:

$$\pi(\theta|y) = \frac{1}{B(a+y, b+n-y)}\theta^{a+y-1}(1-\theta)^{b+n-y-1} = \text{Beta}(a+y, b+n-y).$$

- Recall: If $\theta \sim \text{Beta}(a,b)$, then
  - $\mathbb{E}[\theta] = \frac{a}{a+b}$
  - $\mathbb{V}[\theta] = \frac{ab}{(a+b)^2(a+b+1)}$

Example: rare events

- Step 4. Formulate and state a modeling framework:
  - Under $\text{Beta}(3, 32)$, $\Pr(\theta < 0.1) \approx 0.67$.
  - Posterior distribution for the model:
    $\pi(\theta|Y = y) = \text{Beta}(a + y, b + n - y)$
  - Suppose $Y = 0$. Then, $\pi(\theta|Y = y) = \text{Beta}(3, 32 + 20)$



Example: rare events

- Step 5. Check your models:
  - Compare performance of posterior mean and posterior probability that $\theta < 0.1$.
  - Under $\text{Beta}(3, 52)$,
    - $\Pr(\theta < 0.1|Y = y) \approx 0.92$. More confidence in low values of θ.
    - For $\mathbb{E}(\theta|Y = y)$, we have

$$\mathbb{E}(\theta|y) = \frac{a+y}{a+b+n} = \frac{3}{55} = 0.055.$$

    - Recall that the prior mean is $a/(a+b) = 0.09$. Thus, we can see how that contributes to the prior mean.

$$\begin{aligned}\mathbb{E}(\theta|y) &= \frac{a+b}{a+b+n} \times \text{prior mean} + \frac{n}{a+b+n} \times \text{sample mean} \\ &= \frac{a+b}{a+b+n} \times \frac{a}{a+b} + \frac{n}{a+b+n} \times \frac{y}{n} \\ &= \frac{35}{55} \times \frac{3}{35} + \frac{20}{55} \times \frac{0}{n} = \frac{3}{55} = 0.055.\end{aligned}$$

Example: rare events

- Step 6. Answer the question:
  - People with low prior expectations are generally at least 90% certain that the infection rate is below 0.10.
  - $\pi(\theta|Y)$ is to the left of $\pi(\theta)$ because the observation $Y = 0$ provides evidence of a low value of θ.
  - $\pi(\theta|Y)$ is more peaked than $\pi(\theta)$ because it combines information and so contains more information than $\pi(\theta)$ alone.
  - The posterior expectation is 0.055.
  - The posterior mode is 0.04.
    - Note, for $\text{Beta}(a, b)$, the mode is $(a-1)/(a+b-2)$.
  - The posterior probability that $\theta < 0.1$ is 0.92.

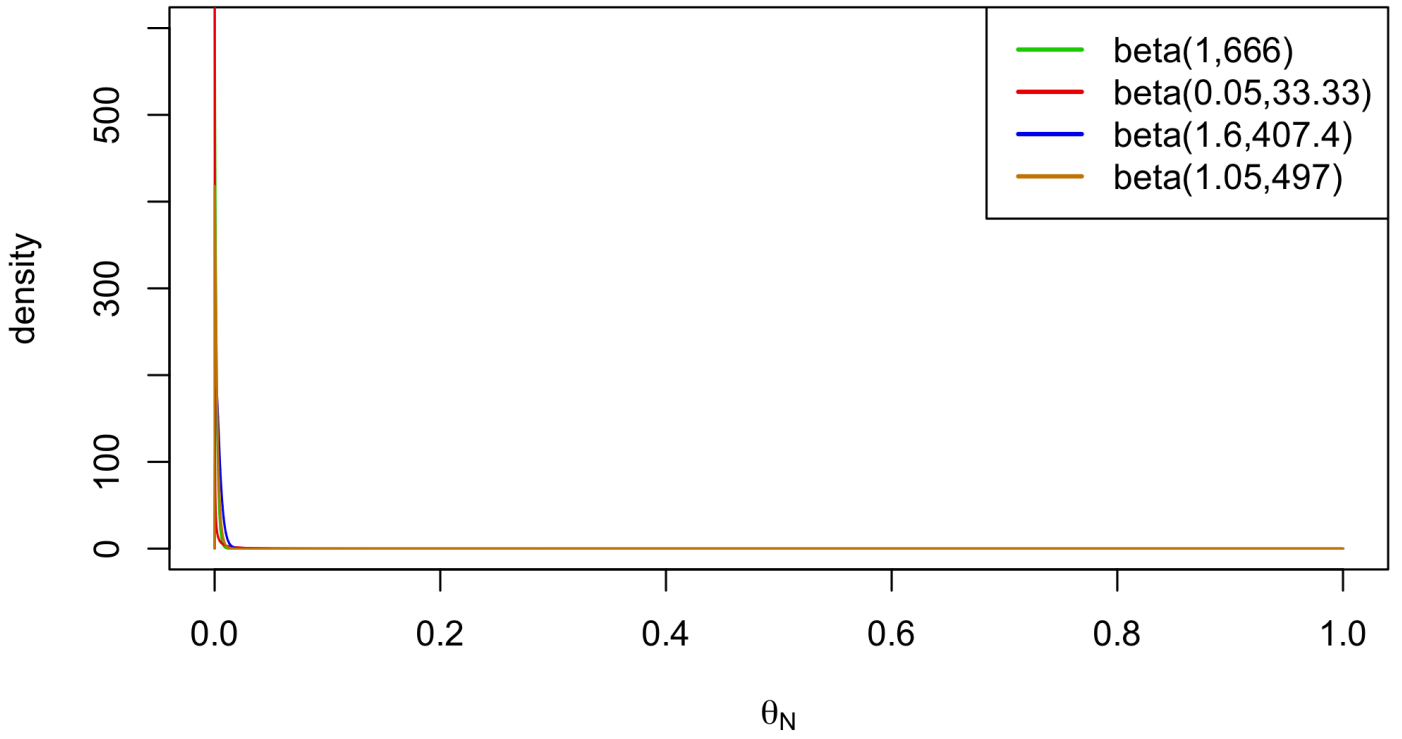
Cautionary tale: parameters at the boundary

- Tuyl et al. (2008) discuss potential dangers of using priors that have $a < 1$ with data that are all 0's (or $b < 1$ with all 1's). They consider data on adverse reactions to a new radiological contrast agent.
- Let θ_N: probability of a bad reaction using the new agent.
- Current standard agent causes bad reactions about 15 times in 10000, so one might think 0.0015 is a good guess for θ_N.
- How do we choose a prior distribution?

Potential prior distributions

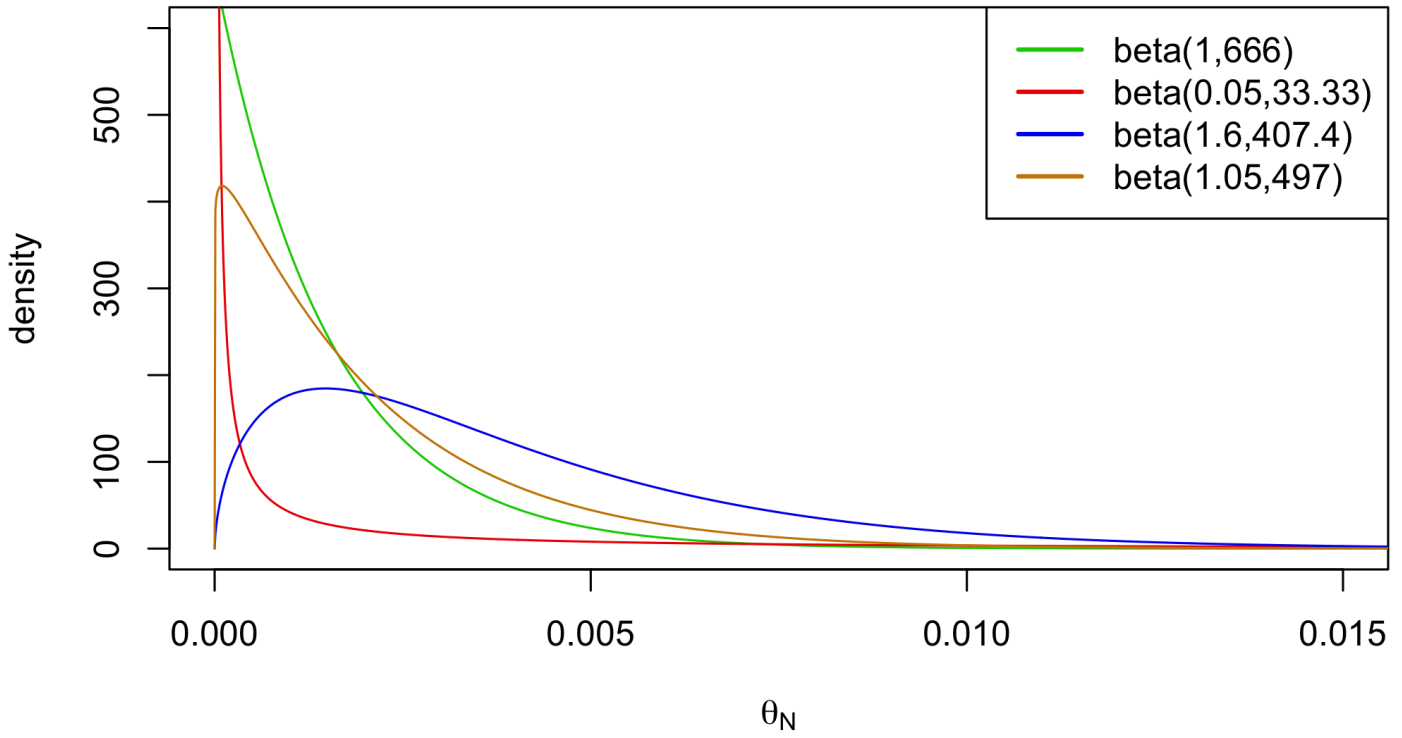
- One might consider a variety of choices centered on $15/10000 = 0.0015$:
  - Prior 1: Beta(1,666) (mean 0.0015; 1 prior bad reaction in 667 administrations)
  - Prior 2: Beta(0.05,33.33) (mean 0.0015; 0.05 prior bad reactions in 33.38 administrations)
  - Prior 3: Beta(1.6, 407.4) (mode 0.0015; 409 prior administrations)
  - Prior 4: Beta(1.05, 497) (median 0.0015; 498.05 prior administrations)
- Does it matter which one we choose?

Potential prior distributions



Potential prior distributions

Let's zoom in:



Potential prior distributions

- Let's take a closer look at properties of these four prior distributions, concentrating on the probability that $\theta_N < 0.0015$.
- That is, new agent not more dangerous than old agent.

	Be(1,666)	Be(0.05,33.33)	Be(1.6,407.4)	Be(1.05,497)
Prior prob	0.632	0.882	0.222	0.500
Post prob (0 events)	0.683	0.939	0.289	0.568
Post prob (1 event)	0.319	0.162	0.074	0.213

- Suppose we have a single arm study of 100 subjects.
- Consider the two most likely potential outcomes:
  - 0 adverse outcomes observed
  - 1 adverse outcome observed

Problems with the priors

- After just 100 trials with no bad reactions, the low weight (33.38 prior observations) prior indicates one should be 94% sure the new agent is equally safe as (or safer than) the old one.
- The low weight prior largely assumes the conclusion we actually hope for (that the new agent is safer), thus it takes very little confirmatory data to reach that conclusion.
- Is this the behavior we want?
- Take home message: be very careful with priors that have $a < 1$ with data that are all 0's (or $b < 1$ with all 1's).
- Given that we know the adverse event rate should be small, we might try a restricted prior e.g. Unif(0,0.1).

What's next?

Move on to the readings for the next module!